# An Observer for Linear Systems with Randomly-Switching Measurement Equations

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**Abstract**

Based on the algebraic transformation of a randomly switched linear measurement equation to a nonlinear, yet deterministic equation, an asymptotic observer is constructed for a class of systems encountered in manufacturing, telecommunications, and embedded control applications. The observer is shown to be locally exponentially stable for any arbitrary switching sequence, and combines the algebraic transformation with the introduction of a Newton observer defined on the resulting nonlinear measurement equation.

## 1 Introduction

The emergence of increasingly complex engineering systems has triggered an intense focus on novel control theoretic areas of research, including sensor and actuator networks, decentralized control, and fault tolerant control. In order for such complex systems to behave in a satisfactory manner, i.e to be subjected to effective control strategies, it is vitally important that the measured sensory data can be incorporated in the control loop under various forms of unreliability. Of particular importance are questions concerning sensor diagnostics and state observability. In this paper, we take the point of view that, in a number of applications, including manufacturing, telecommunications, and embedded systems, sensor failures occur intermittently and go undetected, while no *a priori* probabilities can be given for how prone to failure the sensors are. Instead, we assume that the possible sensory modes of operation are known and, based on this assumption alone, we show how to design local asymptotic observers for linear time-invariant dynamical systems.

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In other words, consider the single-output autonomous system

$$\begin{aligned} x_{k+1} &= Ax_k \\ y_k &= C(\theta_k)x_k, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, and $A$, $C(1), \ldots, C(m)$ are constant matrices of compatible dimensions. Furthermore, $\theta_k$ is an arbitrary mode sequence assuming values in $\{1, \ldots, m\}$, indexing the current measurement equation in such a way that $C(\theta_k)$ switches randomly between $C(1), \ldots, C(m)$, which model the $m$ different sensory modes. It should be stressed again that it is unknown which one of the $m$ different measurement equations is in effect at any given time instant. The goal of this paper can thus be summarized as follows: Devise an asymptotic observer for the system in Equation (1), where the switching sequence $\theta$ is arbitrary and unknown. It appears that this problem has never been successfully addressed in a systematic manner. When the mode sequence $\theta$ is observed, it is well known [4] that a Kalman filter can, under some conditions, be used as an observer for (1). Recently, an LMI-based approach has been proposed for designing Luenberger-like switching observers [1]. For obvious reasons, these results are not pertinent to this work. However, capitalizing on the latter approach and on failure detection techniques, an observer design methodology was proposed in [2]. Unfortunately, failure detection schemes require the parameter $\theta$ to be slowly-varying, which is too restrictive for the problem at hand.

The outline of this paper is as follows: We introduce the novel Direct Algebraic Approach (DAA) in Section 2, and we construct the DAA-Newton observer in Section 3. In Section 4, we show that the DAA-Newton observer results in a local exponential observer for (1). We finally present some numerical results in Section 5.

## 2 The Direct Algebraic Approach (DAA)

In this section, we present the Direct Algebraic Approach, which was originally proposed in [3]. The measurement equation in (1) is equivalent to $y_k - C(i)x_k = 0$ for some $i \in \{1, \ldots, m\}$, namely $i = \theta_k$, which in turn implies that

$$\big(y_k - C(1)x_k\big) \cdots \big(y_k - C(m)x_k\big) = 0. \tag{2}$$

Now, defining

$$g_k(x) \triangleq \big(y_k - C(1)x\big) \cdots \big(y_k - C(m)x\big), \tag{3}$$

we have $g_k(x_k) = 0$. We can therefore shift our attention to designing an observer for the following deterministic system:

$$\begin{array}{l} x_{k+1} = Ax_k \\ g_k(x_k) = 0, \end{array} \tag{4}$$

where $g_k(x_k) = 0$ is the new nonlinear, time-varying, yet deterministic measurement equation. Indeed, $g_k$ is a deterministic polynomial form whose coefficients are determined by the available measurement $y_k$. Clearly, the uncertainty associated with the randomly switched measurement equation of the original system in (1) has been removed. However, the price one has to pay for the introduction of a nonlinear measurement equation is that local convergence is in general all one can hope for. In the next section, we explore the DAA further and combine it with a nonlinear observer to define an observer for our original system in (1).

## 3 The DAA-Newton Observer

In [6], Moraal and Grizzle proposed a nonlinear observer design approach based on Newton's method, which we refer to as the Newton observers approach. It has, so far, been the only approach that could be shown to yield asymptotic observers when combined with the DAA. The key idea is to fix an integer $N_B \geq n$, defined as the "block size", and define a new measurement map as follows:

$$G_k(x) \triangleq \begin{pmatrix} g_k(x) \\ \vdots \\ g_{k+N_B-1}(A^{N_B-1}x) \end{pmatrix} \tag{5}$$

$$= \begin{pmatrix} \prod_{i=1}^{m}(y_k - C(i)x) \\ \vdots \\ \prod_{i=1}^{m}(y_{k+N_B-1} - C(i)A^{N_B-1}x) \end{pmatrix}. \tag{6}$$

Since we have

$$G_k(x_k) = 0, \tag{7}$$

Equation (7) can be used as a new measurement equation, replacing $g_k(x_k) = 0$ in (4) as follows:

$$\begin{array}{l} x_{k+1} = Ax_k \\ G_k(x_k) = 0. \end{array} \tag{8}$$

We can now define the DAA-Newton observer for (1) as:

$$\hat{x}_k^- = A\hat{x}_{k-1} \tag{9}$$

$$\hat{x}_k = \hat{x}_k^- - \big(G_k'(\hat{x}_k^-)\big)^\dagger (G_k(\hat{x}_k^-)), \tag{10}$$

where $G_k'(x)$ is the Jacobian of $G_k(x)$, and where $J^\dagger$ is defined as follows for any $N \times n$ matrix $J$, $N \geq n$:

$$J^\dagger \triangleq (J^T J)^{-1} J^T. \tag{11}$$

Note that $J^\dagger$ exists only when $J$ is of full rank, and in that case coincides with the pseudo inverse of $J$. The observer given by (9-10) is a direct interpretation of the Newton observers approach applied to the system in (8): Equation (10), the "corrector" part of the observer, materializes a single iteration of Newton's method on Equation (7) using $\hat{x}_k^-$ as the initial estimate of the root of $G_k$, exhibiting the "map inversion" viewpoint of [6]. The motivation behind the construction of $G_k$ in (5) thus becomes obvious: Newton's method cannot be shown to converge to $x_k$ if Equation (7) is underdetermined, hence the condition $N_B \geq n$. Note that, by construction of $G_k$, future measurements must be available. This requirement can however be waived if $A$ is invertible, enabling the construction of $G_k$ by stacking past, instead of future, measurement functions $g_k$, which is not pursued in this paper. We now turn to the next section, where we show that the DAA-Newton observer (9-10) results in a local exponential observer for (1).

## 4 Stability

We define a path $\theta$ of length $N$ as a mode string $\theta_1\theta_2 \ldots \theta_N$ with values in $\{1, \ldots, m\}$, and $\Theta_N$ as the set of all $m^N$ paths of length $N$. Given a path $\theta$, we further define $\theta_{[i,j]}$ as the path $\theta_i \ldots \theta_j$. The observability matrix $\mathcal{O}(\theta)$ of a path $\theta$ of length $N$ is defined as:

$$\mathcal{O}(\theta) \triangleq \begin{pmatrix} C(\theta_1) \\ \vdots \\ C(\theta_N)A^{N-1} \end{pmatrix}. \tag{12}$$

We also define the function $\mathcal{P}$ of a pair of paths $\theta_1$ and $\theta_2$ as follows:

$$\mathcal{P}(\theta^1, \theta^2) \triangleq \mathcal{O}(\theta^1) - \mathcal{O}(\theta^2). \tag{13}$$

We now make the following assumption for further analysis:

**Assumption 1** *Given the system in (1), assume that there exist two integers $N_1 \geq n$ and $N_2 \geq n$ such that, letting $N = N_1 + N_2 - 1$, we have:*

1. *For any path $\theta \in \Theta_N$, every $N_1 \times n$ submatrix of $\mathcal{O}(\theta)$ has full rank.*
2. *For any pair of paths $\theta^1$ and $\theta^2$ in $\Theta_N$, if $\theta_i^1 \neq \theta_i^2$ $\forall i \in \{1, \ldots, N\}$, then every $N_2 \times n$ submatrix of $\mathcal{P}(\theta^1, \theta^2)$ has full rank.* $\diamondsuit$

As barbaric as it may seem, Assumption 1 can be shown to be generically satisfied with $N_1 = N_2 = n$ and $N = 2n - 1$. However, it is unknown whether it is decidable. In any case, we can now state the main stability result:

**Theorem 1** *Assume that the system in (1) satisfies Assumption 1 and that $A$ is invertible. Then, whenever $x_0 \neq 0$, the DAA-Newton observer (9-10) results in a local exponential observer for (1) if $N_B = N$.* $\diamondsuit$

We now embark on proving Theorem 1 through a series of lemmas, leading to the proof in section 4.3. In the remainder of the paper, the norm $\|\cdot\|$ is assumed to be the Euclidean (or induced Euclidean) norm, and $B(x, r)$ denotes the open ball of radius $r$ centered on $x$. The $p^{th}$ differential of a function $G$ is written $G^{\{p\}}$, but we will also write $G' = G^{\{1\}}$ and $G'' = G^{\{2\}}$. We also assume that $N_B = N$.

### 4.1 Newton Observers

Consider the nonlinear system

$$\begin{array}{l} x_{k+1} = f(x_k) \\ G_k(x_k) = 0, \end{array} \tag{14}$$

where the measurement map $G_k$ is time-varying and square or overdetermined (i.e. $G_k : \mathbb{R}^n \rightarrow \mathbb{R}^N$, $N \geq n$), so that we can define a Newton observer for (14) as follows:

$$\hat{x}_k^- = f(\hat{x}_{k-1}) \tag{15}$$
$$\hat{x}_k = \mathcal{N}_k(\hat{x}_k^-), \tag{16}$$

where $\mathcal{N}_k$ is given by:

$$\mathcal{N}_k(x) \triangleq x - (G_k'(x))^\dagger G_k(x). \tag{17}$$

In [6], the observer (15-16) was shown to be locally exponential for time-invariant systems that are controlled-invariant with respect to a compact set. We hereby present an extension of that result to time-varying, possibly unstable autonomous systems described by (14). Furthermore, the following result does not require the strong observability assumptions of [6]. As can be anticipated, we will prove Theorem 1 by showing that the system in (8) satisfies the requirements of the following lemma, whose proof is given in Appendix A:

**Lemma 1** *Consider the system in (14). First, assume that $f$ and $G_k$, $k \geq 0$, are in $\mathcal{C}^3(\mathbb{R}^n)$, and that $f$ is globally L-Lipschitz. Furthermore, assume that given $x_0 \in \mathbb{R}^n$, there exists a sequence $R_k$ of subsets of $\mathbb{R}^n$ such that:*

1. *$x_k \in R_k$, $k \geq 0$,*
2. *defining $d_k$ as $\operatorname{dist}(x_k, \overline{R_k})$, where $\overline{R_k}$ is the complement of $R_k$ in $\mathbb{R}^n$, there exists $\beta > 0$ such that $d_{k+1} \geq \beta d_k > 0$,*
3. *and finally,*
   - *(a) $\exists\, g_p > 0$, $\gamma_p > 0$ such that $\sup_{x \in R_k} \|G_k^{\{p\}}(x)\| \leq g_p \gamma_p^k$, $p \in \{1, 2, 3\}$,*
   - *(b) $\exists\, g_\dagger > 0$, $\gamma_\dagger > 0$ such that $\|(G_k'(x_k))^\dagger\| \leq g_\dagger \gamma_\dagger^k$.*

*Then there exist $c > 0$ and $\nu > 0$ such that $\frac{1}{2} \sup_{x \in X_k} \|\mathcal{N}_k''(x)\| \leq c\nu^k$, where $X_k = \{x \in R_k \mid \|(G_k'(x))^\dagger\| \leq 2 g_\dagger \gamma_\dagger^k\}$, and, moreover, the observer given by (15-16) results in a local exponential observer for (14), in the sense that if $\hat{x}_0^-$ satisfies:*

$$\begin{aligned} \|\hat{x}_0^- - x_0\| &\leq \delta, \quad \text{then} && (18) \\ \|\hat{x}_{k+1} - x_{k+1}\| &\leq \alpha \|\hat{x}_k - x_k\|, && (19) \end{aligned}$$

*for all $k \geq 0$, whenever $\alpha$ and $\delta$ satisfy:*

- $0 < \alpha \leq \min\left\{\beta, \frac{1}{\gamma_\dagger \gamma_2}, \frac{1}{\gamma_\dagger^2 \gamma_1 \gamma_2}, \frac{\beta}{\gamma_\dagger \gamma_1}, \frac{1}{\nu}\right\}$,
- $0 \leq \delta < \min\left\{\frac{d_0}{2}, \frac{1}{4 g_\dagger g_2}, \frac{1}{8 g_\dagger^2 g_1 g_2}, \frac{d_0}{4 g_\dagger g_1}, \frac{\alpha}{cL}\right\}$. $\diamondsuit$

Note that our definition of a local exponential observer does not imply that the rate $\alpha$ and radius $\delta$ of convergence are uniform over the entire state space.

### 4.2 Bounding $\|(G_k'(x_k))^\dagger\|$

Note that condition 3.(b) in Lemma 1 requires an upper bound on $\|(G_k'(x_k))^\dagger\|$. In this section, we show that $(G_k'(x_k))^\dagger$ exists and we evaluate an upper bound on its norm. $G_k'(x_k)$ can be expressed as follows:

$$G_k'(x_k) = - \begin{pmatrix} \left(\prod_{j \neq \theta_k} (C(\theta_k) - C(j)) x_k\right) C(\theta_k) \\ \vdots \\ \left(\prod_{j \neq \theta_{k+N-1}} (C(\theta_{k+N-1}) - C(j)) A^{N-1} x_k\right) \\ \times C(\theta_{k+N-1}) A^{N-1} \end{pmatrix} \tag{20}$$

Defining the parameterized function $\mathcal{G}_{\theta,x^*}$ as follows:

$$\mathcal{G}_{\theta,x^*}(x) \triangleq \begin{pmatrix} \prod_{i=1}^{m}(C(\theta_1)x^* - C(i)x) \\ \vdots \\ \prod_{i=1}^{m}(C(\theta_N)A^{N-1}x^* - C(i)A^{N-1}x) \end{pmatrix}, \quad (21)$$

we note that $G_k(x) = \mathcal{G}_{\theta_{[k,k+N_B-1]},x_k}(x)$. Therefore,

$$\|(G'_k(x_k))^\dagger\| \leq \max_{\theta' \in \Theta_N} \|(\mathcal{G}'_{\theta',x_k}(x_k))^\dagger\|, \quad (22)$$

and since $\theta_k \ldots \theta_{k+N-1}$ is unknown, it is natural to bound the right hand side of (22). In fact, the sole purpose of Assumption 1 is to allow for such a bound to be established through the following lemma:

**Lemma 2** *Assume that the system in (1) satisfies Assumption 1. It follows that*

$$\max_{\theta \in \Theta_k} \sup_{\|x\|=1} \|(\mathcal{G}'_{\theta,x}(x))^\dagger\| \quad (23)$$

*is finite.* ♢

**Proof:** We define $\sigma$ as:

$$\sigma \triangleq \min_{\theta \in \Theta_N} \inf_{\|x\|=1} \inf_{\|t\|=1} \|(\mathcal{G}'_{\theta,x}(x))t\|, \quad (24)$$

and we note that if $\sigma > 0$, then

$$\max_{\theta \in \Theta_N} \sup_{\|x\|=1} \|(\mathcal{G}'_{\theta,x}(x))^\dagger\| = \frac{1}{\sigma}, \quad (25)$$

and is finite. Therefore, we need to show that $\sigma$ is positive. We define the scalars $\rho$ and $\psi$ as follows:

$$\rho \triangleq \min_{\theta \in \Theta} \inf_{\|t\|=1} \min_{\phi \in \Phi} \|\phi(\mathcal{O}(\theta))t\|, \quad (26)$$

$$\psi \triangleq \min_{\theta \in \Theta} \inf_{\|x\|=1} \min_{u \in U} \max_{i=1,\ldots,N_2} \prod_{j \neq u(i)} |(C(\theta_{u(i)}) - C(j))A^{u(i)-1}x|, \quad (27)$$

where $\Phi$ is the set of all functions $\phi$ that extract an $N_1 \times n$ submatrix from a matrix $\mathcal{O}(\theta)$, and where $U$ is the set of all maps $u : \{1, \ldots, N_2\} \to \{1, \ldots, N\}$. Clearly, $\sigma \geq \psi\rho$. It therefore suffices to show that both $\rho$ and $\psi$ are positive. $\rho > 0$ follows directly from condition 1. Now, suppose that $\psi = 0$. Then there exist a path $\theta$, $x \in \mathbb{R}^n$ ($\|x\| = 1$), and $u \in U$ such that

$$\max_{i=1,\ldots,N_2} \prod_{j \neq u(i)} |(C(\theta_{u(i)}) - C(j))A^{u(i)-1}x| = 0, \quad (28)$$

which implies that

$$\prod_{j \neq u(i)} |(C(\theta_{u(i)}) - C(j))A^{u(i)-1}x| = 0, \quad (29)$$

$\forall i \in \{1, \ldots, N_2\}$, which contradicts condition 2. □

### 4.3 Proof of Theorem 1

The proof lies in showing that the system in (8) satisfies the requirements for Lemma 1, under the assumptions of Theorem 1. First of all, the dynamics being linear and $G_k$ being polynomial in the state, they are both in $\mathcal{C}^3(\mathbb{R}^n)$. Moreover, $A$ being invertible, there exist $l > 0$ and $L > 0$ such that

$$l\|x\| \leq \|Ax\| \leq L\|x\|, \quad (30)$$

$\forall x \in \mathbb{R}^n$. This implies that the dynamics is $L$-Lipschitz.

Next, since $x_0 \neq 0$, there exist $r_0 > 0$, $r'_0 > 0$ such that $r_0 < \|x_0\| < r'_0$. Letting $r_k = r_0 l^k$ and $r'_k = r'_0 L^k$, we get $x_k \in R_k$, $k \geq 0$, where $R_k \triangleq \{x \in \mathbb{R}^n \mid r_k < \|x\| < r'_k\}$. Clearly, $d_{k+1} \geq l d_k > 0$. It now remains to prove that conditions 3.(a) and 3.(b) in Lemma 1 are met.

To prove that condition 3.(a) is met, we note that for $p \in \{1, 2, 3\}$,

$$\sup_{x \in R_k} \|G_k^{\{p\}}(x)\| \leq \max_{\theta' \in \Theta_N} \sup_{x^* \in R_k} \sup_{x \in R_k} \|\mathcal{G}^{\{p\}}_{\theta',x^*}(x)\|. \quad (31)$$

Since $r'_k = r'_0 L^k$, and since $\mathcal{G}^{\{p\}}_{\theta,x^*}(x)$ is polynomial in $x^*$ and $x$, it is straightforward to show that there exist $g_p > 0$ and $\gamma_p > 0$, $p \in \{1, 2, 3\}$, such that

$$\max_{\theta \in \Theta_N} \sup_{\|x^*\| \leq r'_k} \sup_{\|x\| \leq r'_k} \|\mathcal{G}^{\{p\}}_{\theta,x^*}(x)\| \leq g_p \gamma_p^k, \quad (32)$$

for $k \geq 0$, and since $R_k \subset \{x \mid \|x\| \leq r'_k\}$, we get

$$\sup_{x \in R_k} \|G_k^{\{p\}}(x)\| \leq g_p \gamma_p^k. \quad (33)$$

As for condition 3.(b), we recall that

$$\|(G'_k(x_k))^\dagger\| \leq \max_{\theta' \in \Theta_N} \sup_{x \in R_k} \|(\mathcal{G}'_{\theta',x}(x))^\dagger\|. \quad (34)$$

By Lemma 2, we have that

$$\frac{1}{\sigma} = \max_{\theta \in \Theta_N} \sup_{\|x\|=1} \|(\mathcal{G}'_{\theta,x}(x))^\dagger\| \quad (35)$$

is finite. Consequently,

$$\max_{\theta \in \Theta_N} \sup_{x \in R_k} \|(\mathcal{G}'_{\theta,x}(x))^\dagger\| = \max_{\theta \in \Theta_N} \sup_{\|x\|=r_k} \|(\mathcal{G}'_{\theta,x}(x))^\dagger\| \quad (36)$$

$$= \frac{\max_{\theta \in \Theta_N} \sup_{\|x\|=1} \|(\mathcal{G}'_{\theta,x}(x))^\dagger\|}{r_k^{m-1}} \quad (37)$$

$$= \frac{1}{\sigma r_k^{m-1}}, \quad (38)$$

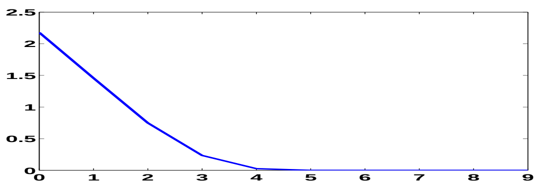
hence 3.(b) with $g_\dagger = \frac{1}{\sigma r_0^{m-1}}$ and $\gamma_\dagger = \frac{1}{l^{m-1}}$. □

## 5 Numerical Results

In this section, we evaluate the performance of the DAA-Newton observer (9-10) by numerical simulation. Let us consider the following system:

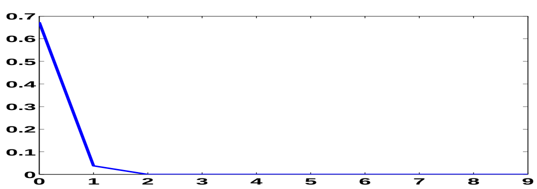
$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \left\{ \begin{array}{lcl} C(1) & = & (1 \quad 0) \\ C(2) & = & (2 \quad 3) \end{array} \right. , \tag{39}$$

which satisfies the assumptions of Theorem 1 (with $N_1 = 2$, $N_2 = 2$ and $N = 3$). The radius of convergence is evaluated by numerical simulation to be $\simeq 0.4$ at $x_0 = (1,1)^T$. In Figure 1, the observer error $\|\hat{x}_k - x_k\|$ is plotted versus time for $\hat{x}_0^- = (0.7, 0.7)^T$ and for three different mode sequences $\theta$, demonstrating the stability of the observer.




(a) $\theta = 1 \cdot 1 \cdot 1 \cdot 1 \cdots$


(b) $\theta = 2 \cdot 2 \cdot 2 \cdot 2 \cdots$




(c) $\theta = 1 \cdot 2 \cdot 1 \cdot 2 \cdots$

**Figure 1:** DAA-Newton observer error versus time for different mode sequences $\theta$.

## 6 Conclusion

An observer design approach has been presented for linear discrete-time systems with randomly-switching measurement equations. It has been shown to yield local exponential observers, and numerical simulation supports the soundness of the approach. Given the current results, the approach is being evaluated on more general systems, such as nonautonomous and multi-output systems. Moreover, using the DAA to design estimators for noisy systems is currently under investigation.

## A Proof of Lemma 1

We first state the following standard result (adapted from [5, pp 279-281 & p 309]), which establishes the convergence of Newton's method:

**Theorem 2** *Let $G$ be a mapping from $\mathbb{R}^n$ to $\mathbb{R}^N$, where $N \geq n$, and assume that $G$ is three times continuously differentiable. Assume further that:*

1. *There is a point $x_1 \in X$ such that $(G'(x_1))^\dagger$ exists with $\|(G'(x_1))^\dagger\| \leq \beta$ and $\|(G'(x_1))^\dagger G(x_1)\| \leq \eta$.*
2. *There exists $r \geq 2\eta$ such that $\sup_{x \in T} \|G''(x)\| \leq K$, where $T = B(x_1, r)$.*
3. *The constant $h = \beta\eta K$ satisfies $h < \frac{1}{2}$.*

*Then the sequence $x_{n+1} = \mathcal{N}(x_n) \triangleq x_n - (G'(x_n))^\dagger G(x_n)$ of successive approximations generated by Newton's method exists for all $n \geq 1$, remains in $T$, and converges to a solution of $G(x) = 0$. Moreover, the rate of convergence is given by*

$$\|x_{n+1} - x^*\| \leq \mu \|x_n - x^*\|^2, \tag{40}$$

*where* $\mu = \frac{1}{2} \sup\limits_{x \in T} \|\mathcal{N}''(x)\|$. $\diamondsuit$

We now establish the lemma.

**Proof of Lemma 1:** First, $\|\mathcal{N}_k''(x)\|$ needs to be adequately bounded. Schematically, note that for scalar $\mathcal{N}$, $G$ and $x$, we have $\mathcal{N}'' = \frac{(G')^3 G'' - 2GG'(G'')^2 + G(G')^2 G'''}{(G')^4}$. This shows that $\mathcal{N}_k''(x)$ is polynomial in $G_k^{\{p\}}(x)$, $p \in \{1,2,3\}$, and $(G_k'(x))^\dagger$. Since these terms are bounded by exponentials over $X_k$, it is straightforward to bound the polynomial by an exponential, finding $c > 0$ and $\nu > 0$ such that $\frac{1}{2} \sup\limits_{x \in X_k} \|\mathcal{N}_k''(x)\| \leq c\nu^k$, $k \geq 0$.

We now show by induction on $k$ that

$$\|\hat{x}_k - x_k\| \quad \leq \quad \frac{\alpha}{L} \|\hat{x}_k^- - x_k\|, \qquad k \geq 0, \tag{41}$$

and note that Equation (41), combined with the fact that $f$ is globally $L$-Lipschitz, yields:

$$\|\hat{x}_{k+1} - x_{k+1}\| \quad \leq \quad \alpha \|\hat{x}_k - x_k\|, \qquad k \geq 0, \tag{42}$$

which establishes (19). Note that we also get $\|\hat{x}_{k+1}^- - x_{k+1}\| \leq \alpha \|\hat{x}_k^- - x_k\|$.

Equation (41) for $k = 0$ is a direct consequence of Lemma 3 and of Equation (18).

Now, assume that Equation (41) is true up to time $k = K-1$, or in other words that $\|\hat{x}_k - x_k\| \leq \frac{\alpha}{L}\|\hat{x}_k^- - x_k\|$ for $0 \leq k \leq K-1$. Since $f$ is globally $L$-Lipschitz, we furthermore have that $\|\hat{x}_{k+1}^- - x_{k+1}\| \leq L\|\hat{x}_k - x_k\|$ for $0 \leq k \leq K-1$. Combining these last two facts, we get

$$\|\hat{x}_K^- - x_K\| \leq \alpha^K \|\hat{x}_0^- - x_0\| \leq \alpha^K \delta, \tag{43}$$

which, again by Lemma 3, establishes Equation (41) for $k = K$. □

**Lemma 3** *If* $\|\hat{x}_k^- - x_k\| \leq \alpha^k \delta$, *then* $\|\hat{x}_k - x_k\| \leq \frac{\alpha}{L}\|\hat{x}_k^- - x_k\|$. ♢

**Proof:** We first define, for $k \geq 0$:

- $\beta_k = \|(G_k'(\hat{x}_k^-))^\dagger\|$,
- $\eta_k = \|(G_k'(\hat{x}_k^-))^\dagger G_k(\hat{x}_k^-)\|$,
- $h_k = \beta_k \eta_k g_2 \gamma_2^k$,
- $\rho_k = \min\left\{d_k, \frac{1}{2g_\dagger \gamma_\dagger^k g_2 \gamma_2^k}\right\}$, and $S_k = B(x_k, \rho_k)$,
- $\mu_k = \frac{1}{2}\sup_{T_k} \|\mathcal{N}_k''(x)\|$, where $T_k = B(\hat{x}_k^-, 2\eta_k)$, and we note that $\mu_k \leq c\nu^k$ if $T_k \subset X_k$.

We next note that since $\delta < \min\left\{\frac{d_0}{2}, \frac{1}{4g_\dagger g_2}\right\}$ and $\alpha \leq \min\left\{\beta, \frac{1}{\gamma_\dagger \gamma_2}\right\}$, we have that $\alpha^k \delta < \frac{\rho_k}{2}$, and that $\hat{x}_k^- \in S_k$. Moreover, $S_k \subset R_k$ because $\rho_k \leq d_k$. Therefore, by Lemma 4:

$$\sup_{x \in S_k} \|(G_k'(x))^\dagger\| \leq 2g_\dagger \gamma_\dagger^k. \tag{44}$$

Thus, since $\hat{x}_k^- \in S_k \subset R_k$, $\|(G_k'(\hat{x}_k^-))^\dagger\| \leq g_\dagger \gamma_\dagger^k$, which implies that $\beta_k \leq g_\dagger \gamma_\dagger^k$ and $\eta_k \leq \|(G_k'(\hat{x}_k^-))^\dagger\| \cdot \|G_k(\hat{x}_k^-)\| \leq g_\dagger \gamma_\dagger^k \|G_k(\hat{x}_k^-)\| \leq g_\dagger \gamma_\dagger^k g_1 \gamma_1^k \|\hat{x}_k^- - x_k\| \leq g_\dagger \gamma_\dagger^k g_1 \gamma_1^k \alpha^k \delta$. Therefore, $h_k = \beta_k \eta_k g_2 \gamma_2^k \leq \delta g_\dagger^2 g_1 g_2 \alpha^k \gamma_\dagger^{2k} \gamma_1^k \gamma_2^k$, and since $\delta < \frac{1}{8g_\dagger^2 g_1 g_2} < \frac{1}{2g_1 g_\dagger^2 g_2}$ and $\alpha \leq \frac{1}{\gamma_\dagger^2 \gamma_1 \gamma_2}$, we get

$$h_k < \frac{1}{2}. \tag{45}$$

We now consider the open ball $T_k$. From $\delta < \min\left\{\frac{1}{8g_\dagger^2 g_1 g_2}, \frac{d_0}{4g_\dagger g_1}\right\}$ and $\alpha \leq \min\left\{\frac{1}{\gamma_\dagger^2 \gamma_1 \gamma_2}, \frac{\beta}{\gamma_\dagger \gamma_1}\right\}$, we get that $2\eta_k \leq 2g_\dagger \gamma_\dagger^k g_1 \gamma_1^k \alpha^k \delta < \frac{\rho_k}{2}$, which, given that $\alpha^k \delta < \frac{\rho_k}{2}$, implies that $T_k \subset S_k \subset R_k$. Therefore,

$$\sup_{x \in T_k} \|G_k''(x)\| \leq g_2 \gamma_2^k. \tag{46}$$

Finally, since $2\eta_k \geq \frac{1}{h_k}\left(1 - \sqrt{1 - 2h_k}\right)\eta_k$, then by virtue of Theorem 2 and of Equations (45) and (46), Newton's method would converge to a solution of $G_k(x) = 0$ inside $T_k$. By virtue of Lemma 4 and of the fact that $\rho_k = \min\left\{d_k, \frac{1}{2g_\dagger \gamma_\dagger^k g_2 \gamma_2^k}\right\}$, $x_k$ is the unique solution of $G_k(x) = 0$ in $S_k$. Since $T_k \subset S_k$, Newton's method converges to $x_k$, and we get, from Equation (40), that

$$\|\hat{x}_k - x_k\| \quad \leq \quad \mu_k \|\hat{x}_k^- - x_k\|^2, \tag{47}$$

and since $T_k \subset X_k$ (thanks to the fact that $T_k \subset S_k \subset R_k$ and to Equation (44)), we have that $\mu_k \leq c\nu^k$, which, combined with $\delta < \frac{\alpha}{cL}$, $\alpha \leq \frac{1}{\nu}$, and Equation (47), implies that

$$\|\hat{x}_k - x_k\| \leq c\nu^k \alpha^k \delta \|\hat{x}_k^- - x_k\| \leq \frac{\alpha}{L}\|\hat{x}_k^- - x_k\|, \tag{48}$$

which completes the proof. □

The following straightforward lemma is provided without proof:

**Lemma 4** *Let* $G : R \to \mathbb{R}^N$, *where* $R$ *is an open subset of* $\mathbb{R}^n$ *and* $N \geq n$. *Assume that* $G \in \mathcal{C}^2(R)$, *and that there exists* $x^* \in R$ *such that* $G(x^*) = 0$. *Assume further that there exist two positive scalars* $g_\dagger$ *and* $g_2$ *such that* $\|(G'(x^*))^\dagger\| \leq g_\dagger$ *and* $\sup_{x \in R} \|(G''(x))\| \leq g_2$, *and let* $r \leq \min\left\{\operatorname{dist}(x^*, \overline{R}), \frac{1}{2g_\dagger g_2}\right\}$. *then* $\sup_{x \in B(x^*, r)} \|(G'(x))^\dagger\| \leq 2g_\dagger$, *and, moreover,* $x^*$ *is the unique solution of* $G(x) = 0$ *in* $B(x^*, r)$. ♢

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