Solutions to Homework Set 4: ECE6550
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1

a

\[ \Gamma = \begin{bmatrix} B & AB \end{bmatrix}, \]

where

\[ AB = \begin{bmatrix} \alpha \\ \beta + 1 \end{bmatrix}. \]

And hence

\[ \Gamma = \begin{bmatrix} 1 & \alpha \\ 1 & \beta + 1 \end{bmatrix}, \]

which has \( \text{rank}(\Gamma) = 2 \) as long as \( \alpha \neq \beta + 1 \). Hence the system is completely controllable.

b

Let \( K = [k_1, k_2] \), which gives the closed-loop system

\[ A - BK = \begin{bmatrix} \alpha & 0 \\ \beta & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} \alpha - k_1 & -k_2 \\ \beta - k_1 & 1 - k_2 \end{bmatrix}. \]

The characteristic polynomial is

\[ \chi_{A-BK}(\lambda) = \text{det}(\lambda I - (A - BK)) = \lambda^2 + \lambda(k_2 + k_1 - \alpha - 1) + (\beta - \alpha)k_2 + \alpha - k_1. \]

We want the poles in \( -1 \pm j \) and hence the desired characteristic polynomial is

\[ \varphi(\lambda) = (\lambda + 1 + j)(\lambda + 1 - j) = \lambda^2 + 2\lambda + 2. \]

Identification of the coefficients gives

\[ \begin{cases} k_1 + k_2 - 1 - \alpha = 2 \\ (\beta - \alpha)k_2 + \alpha - k_1 = 2 \end{cases} \Rightarrow \begin{cases} k_1 = 3 + \alpha + \frac{5}{\beta + 1 - \alpha} \\ k_2 = \frac{5}{\beta + 1 - \alpha}. \end{cases} \]

2

a

We have

\[ \Omega = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \alpha & 0 \end{bmatrix}. \]

Hence \( \text{rank}(\Omega) = 1 \) and the system is not completely observable.
Let \( u = -Ly = -LCx \), where \( L \in \mathbb{R} \). We have
\[
A - BLC = \begin{bmatrix} \alpha - L & 0 \\ \beta - L & 1 \end{bmatrix},
\]
with characteristic polynomial
\[
\chi_{A - BLC}(\lambda) = \lambda^2 + \lambda (L - \alpha - 1) + \alpha - L.
\]
Identification of the coefficients gives
\[
\begin{aligned}
L - \alpha - 1 &= 2 \\
\alpha - L &= 2
\end{aligned}
\]
\[
\Rightarrow \begin{aligned}
L &= 3 + \alpha \\
L &= -2 + \alpha,
\end{aligned}
\]
which is impossible. Hence output feedback doesn’t work in this case.

3
This is a silly question. Since the system is completely controllable we already have a controllable decomposition, with
\[
A = \begin{bmatrix} \alpha & 0 \\ \beta & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.
\]

4
We first need to establish a basis for \( \mathcal{N}(\Omega) \) and for \( \mathcal{N}(\Omega)^\perp \).
\[
\Omega x = \begin{bmatrix} x_1 \\ \alpha x_1 \end{bmatrix} = 0 \Rightarrow x_1 = 0.
\]
Hence
\[
\mathcal{N}(\Omega) = \text{span} \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right), \quad \mathcal{N}(\Omega)^\perp = \text{span} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).
\]
What this means is that the transformation matrix \( T \) is equal to the identity matrix and the system is already given by the observable decomposition. Hence the answer is the same as for Question 3.

5
a
We note that since \( \text{rank}(\Gamma) = 2 \) we have that
\[
\mathcal{R}(\Gamma) = \text{span} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right).
\]
Hence
\[
\mathcal{R}(\Gamma) \cap \mathcal{N}(\Omega)^\perp = \text{span} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right),
\]
\[
\mathcal{R}(\Gamma) \cap \mathcal{N}(\Omega) = \text{span} \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right),
\]
\[
\mathcal{R}(\Gamma)^\perp \cap \mathcal{N}(\Omega)^\perp = \{0\},
\]
\[
\mathcal{R}(\Gamma)^\perp \cap \mathcal{N}(\Omega) = \{0\}.
\]
What this means is again that \( T = I \) and again we already have the Kalman decomposition. Hence the answer is the same as for Questions 3 and 4.
b

The McMillan degree is given by the dimension of the intersection between the controllable and observable subspaces. And since

$$\dim(\mathcal{R}(\Gamma) \cap \mathcal{N}(\Omega)^\perp) = 1$$

we have that the McMillan degree is 1.