Sample Midterm: ECE6550
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Closed books, closed notes, closed calculator exam.
Exam time: 80 minutes.

This exam gives a maximum of 50 points. (50 points/question.) For maximum credit, be explicit about the different steps in your solution.

1
Let
\[ \dot{x}(t) = \sin(t)x(t) + t^2, \quad x(0) = 1. \]
What is \( x(1) \)? (You don’t have to solve \( x(1) \) explicitly. Just write down an expression for the solution.)

2
Let
\[
A = \begin{bmatrix}
3 & 0 & 1 \\
1 & 1 & 0 \\
-4 & -1 & -1 \\
\end{bmatrix}.
\]
a
What is \( \mathcal{R}(A) \)?

b
What is \( \mathcal{N}(A) \)?

c
What is \( \mathcal{R}(A)^\perp \)?
3

Consider the phase plot in Figure 1. Which of the following three $A$-matrices was used to generate this plot, with $\dot{x} = Ax$?

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$  

![Figure 1:](image_url)

4

Let

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u.$$

Is it possible to drive this system from $x(0) = (1, 1)^T$ to $x(1) = (2, 0)^T$?

5

Let

$$x_{k+1} = Ax_k.$$

Show that if $|\lambda(A)| < 1$ then

$$P = \sum_{k=0}^{\infty} (A^T)^k Q A^k$$

is a solution to the discrete Lyapunov equation

$$P = A^T P A + Q,$$

given any matrix $Q$.  

SOLUTIONS

1A
We know that for linear, time varying systems we have

\[ x(t) = \Phi(t,0)x(0) + \int_0^t \Phi(t,s)B(s)u(s)ds. \]

In our case we have \( A(t) = \sin(t) \). Moreover, since

\[ \left[ \int_0^t A(s)ds, A(t) \right] = 0 \]

since \( x \) is scalar we have

\[ \Phi(t, t_0) = e^{\int_{t_0}^t \sin(s)ds} = e^{\cos(t_2) - \cos(t_1)}. \]

Moreover, \( u(t) = t^2 \) and \( B(t) = 1 \), which gives

\[ x(1) = e^{1 - \cos(1)} + \int_0^1 e^{\cos(t) - \cos(s)}s^2ds. \]

2A
a

\[ Ax = \begin{bmatrix} 3x_1 + x_3 \\ x_1 + x_2 \\ -4x_1 - x_2 - x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}. \]

We directly see that \( y_1 \) and \( y_2 \) can be whatever, while \( y_3 = -y_1 - y_2 \). Hence

\[ \mathcal{R}(A) = \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right). \]

b

\[ Ax = 0 \Rightarrow \begin{cases} 3x_1 + x_3 = 0 \Rightarrow x_3 = -3x_1 \\ x_1 + x_2 = 0 \Rightarrow x_2 = -x_1 \end{cases} \]

Hence

\[ \mathcal{N}(A) = \text{span} \left( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right). \]

c

Since \( x \) has to be orthogonal to both basis vectors in \( \mathcal{R}(A) \), we have

\[ x^T \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = x_1 - x_3 = 0 \Rightarrow x_1 = x_3, \]

and

\[ x^T \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = x_2 - x_3 = 0 \Rightarrow x_2 = x_3. \]
In other words,

\[ \mathcal{R}(A) = \text{span} \left( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right). \]

3A

Since the plot in the figure corresponds to \( x = 0 \) being an unstable equilibrium we need to see which of the \( A \)-matrices is consistent with this. We have

\[ \lambda(A_1) = \{-1, -1\}, \quad \lambda(A_2) = \{-1, 0\}, \quad \lambda(A_3) = \{1, -1\}, \]

and hence \( A_3 \) was used to generate the plot since it has a positive eigenvalue.

4A

The reachability matrix is

\[ \Gamma = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \]

And, since \( \text{rank}(\Gamma) = 2 \) the system is completely controllable and hence we can go from any point to any other point.

5A

We know that the sum converges if \( |\lambda(A)| < 1 \). Hence \( P \) is well-defined. Moreover,

\[ A^T P A = \sum_{k=0}^{\infty} (A^T)^{k+1} Q A^{k+1} = \sum_{k=0}^{\infty} (A^T)^{k} Q A^{k} - (A^T)^{0} Q A^{0} = P - Q. \]